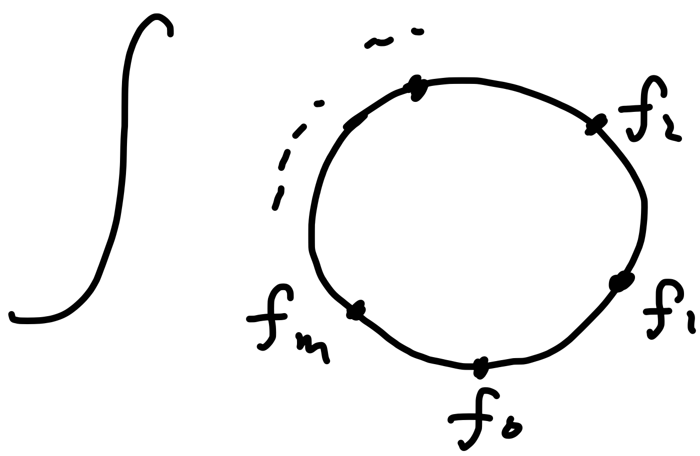
§8. Topological Quantum Mechanics – II

Last time: $\varphi: \Omega^{\bullet}_{S^1} \longmapsto V = \mathbb{R}^{2n}$

$\Rightarrow \langle \cdots \rangle_{free}: \quad C_{-\bullet}(\mathcal{W}_{2n}) \longmapsto \Omega^{-\bullet}_{2n}((\hbar))$

$$b \qquad\qquad \hbar\Delta$$

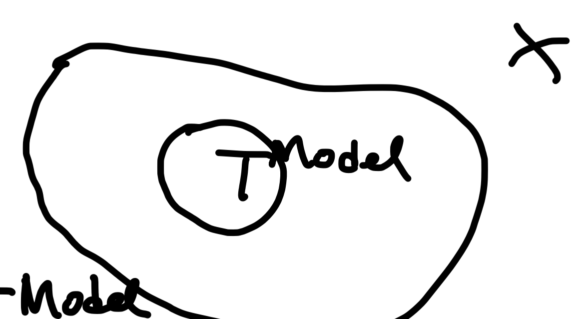
$$B \qquad\qquad d_{2n}$$



Today: Glue this construction to a symplectic mfd.

$\Rightarrow$ Algebraic index

Basic idea: $\Sigma \longmapsto$



- Glue the local model $\Sigma \mapsto T^{\text{Model}}$.

Ref Today: [Gui-L-Xu] Geometry of Localized effective theories, Exact semi-classical approximation and the alg. Index CMP 2021

Gluing via Gelfand–Kazhdan formal geometry

<u>Def'n</u>: A Harish-Chandra pair is a pair (g, K) where g is a Lie algebra, K is a Lie group with

- $K \xrightarrow{\rho} \mathrm{Aut}(g)$, K acts on g
- $\mathrm{Lie}(K) \xhookrightarrow{i} g$

such that they are compatible

$$\begin{tikzcd} \mathrm{Lie}(K) \arrow[r, hook, "i"] \arrow[dr, "d\rho"'] & g \arrow[d, "\text{adjoint}"] \\ & \mathrm{Der}(g) \end{tikzcd}$$

A (g, K)-module is a vector space V w/.

- $K \xrightarrow{\varphi} GL(V)$, K acts on V
- Lie alg. morphism $g \longmapsto \mathrm{End}(V)$

Such that they are compatible

$$\begin{tikzcd} \mathrm{Lie}(K) \arrow[r, hook] \arrow[dr, "d\varphi"'] & g \arrow[d] \\ & \mathrm{End}(V) \end{tikzcd}$$

Def'n. A flat $(\mathfrak{g}, K)$-bundle over X is

- a principal K-bundle $\begin{array}{c} P \\ \downarrow \pi \\ X \end{array}$

- a K-equivariant $\mathfrak{g}$-valued 1-form $\gamma \in \Omega^1(P, \mathfrak{g})$ on P

satisfying

① $\forall\, a \in Lie(K)$, Let $\xi_a \in Vect(P)$ generated by a. Then

$$\gamma(\xi_a) = a$$

$$\begin{tikzcd} 0 \arrow[r] & Lie(K) \arrow[r] \arrow[dr, hook] & Vect(P) \arrow[d, "\gamma"] \\ & & \mathfrak{g} \end{tikzcd}$$

② γ satisfies the Maurer–Cartan equation

$$d\gamma + \frac{1}{2}[\gamma, \gamma] = 0$$

de Rham on P bracket in $\mathfrak{g}$.

Given a flat $(\mathfrak{g},K)$-bundle $\begin{array}{c} P \\ \downarrow \\ X \end{array}$ and $(\mathfrak{g},K)$-module V,

let $\Omega^\cdot(P,V) := \Omega^\cdot(P)\otimes V$

It carries a connection

$$\nabla^\gamma = d+\gamma : \Omega^\cdot(P,V)\to\Omega^{\cdot+1}(P,V)$$

which is flat by the Maurer-Cartan eqn.

Let $V_P := P\times_K V$ associated vector bundle
$$\begin{array}{c} V_P \\ \downarrow \\ X \end{array}$$
on X.

Similar to the usual principal bundle case,

∇^γ induces a flat connection on $\begin{array}{c} V_P \\ \downarrow \\ X \end{array}$

$\Rightarrow$ $\left(\Omega^\cdot(X;V_P),\nabla^\gamma\right)$ chain complex

and $H^\cdot(X,V_P)$ denotes the corresponding cohomology.

Def'n. Let V be a $(\mathfrak{g},K)$-module. Define the $(\mathfrak{g},K)$ relative Lie alg. cochain complex $\left(C^\bullet_{Lie}(\mathfrak{g},K;V), \partial_{Lie}\right)$ by

$$C^p_{Lie}(\mathfrak{g},K;V) = \mathrm{Hom}_K\left(\Lambda^p(\mathfrak{g}/\mathfrak{Lie}(K)), V\right)$$

For $\alpha \in C^p_{Lie}(\mathfrak{g},K;V)$

$$(\partial_{Lie}\alpha)(a_1 \wedge \cdots \wedge a_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i-1} a_i \cdot \alpha(a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_{n+1})$$

$$+ \sum_{i<j} (-1)^{i+j} \alpha([a_i,a_j] \wedge \cdots \hat{a}_i \cdots \hat{a}_j \cdots \wedge a_{n+1})$$

The corresponding cohomology is $H^\bullet_{Lie}(\mathfrak{g},K;V)$

Given a $(\mathfrak{g},K)$-module V and flat $(\mathfrak{g},K)$-bundle $P \xrightarrow{} X$ w/. $\gamma \in \Omega^1(P,\mathfrak{g})$ the flat connection. We can define

$$\mathrm{desc} : \left(C^\bullet_{Lie}(\mathfrak{g},K;V), \partial_{Lie}\right) \longmapsto \left(\Omega^\bullet(X;V_P), \nabla^\gamma\right)$$

$$\alpha \longmapsto \alpha(\gamma, \cdots, \gamma)$$

inducing

$$\mathrm{desc} : H^\bullet_{Lie}(\mathfrak{g},K;V) \longmapsto H^\bullet(X;V_P)$$

· Fedosov connection revisited.

Recall the (formal) Weyl algebras

$$\mathcal{W}_{2n} = \mathbb{R}[[p_i, q^i]]((\hbar)), \quad \mathcal{W}_{2n}^+ = \mathbb{R}[[p_i, q^i]][[\hbar]]$$

w/. induced Lie algebra structure

$$[f, g] := \frac{1}{\hbar}[f, g]_* = \frac{1}{\hbar}(f * g - g * f)$$

We can identify

$$\begin{array}{l} sp_{2n} = \text{quadratic polynomial in } \mathbb{R}[p_i, q^i] \\ \cap \\ \mathcal{W}_{2n}^+ \end{array}$$

$Sp_{2n} \curvearrowright \mathbb{R}^{2n}$ induces $Sp_{2n} \curvearrowright \mathcal{W}_{2n}^+$

$\Rightarrow (\mathcal{W}_{2n}^+, Sp_{2n})$ and $(\mathcal{W}_{2n}, Sp_{2n})$

are Harish-Chandra pairs.

Let (X,ω) be a symplectic mfd. $F_{Sp}(X)$ the symplectic frame bundle. We have the Weyl bundles

$$\mathcal{W}_X^+ = F_{Sp}(X) \underset{Sp_{2n}}{\times} \mathcal{W}_{2n}^+ \qquad \mathcal{W}_X = F_{Sp}(X) \underset{Sp_{2n}}{\times} \mathcal{W}_{2n}$$

Consider the following Harish-Chandra pair

$$(\bar{\mathfrak{g}}, K) = (\mathfrak{g}/Z(\mathfrak{g}), Sp_{2n})$$

where $\mathfrak{g} = \mathcal{W}_{2n}^+$, $Z(\mathfrak{g}) = \mathbb{R}((\hbar))$.

Fedosov $\Longrightarrow$ flat $(\bar{\mathfrak{g}}, K)$-bundle
$$\begin{array}{c} F_{Sp}(X) \\ \downarrow \\ X \end{array}$$

and $H^0(X; \mathcal{W}_X^+)$ gives a deformation quantization.

Choose the trivial $(\bar{\mathfrak{g}}, K)$-module $\mathbb{R}((\hbar))$. Then

$$\text{desc}: \quad C^\bullet_{Lie}(\overline{\mathcal{W}_{2n}^+}, sp_{2n}; \mathbb{R}((\hbar))) \cong C^\bullet_{Lie}(\mathcal{W}_{2n}^+, sp_{2n} \oplus Z(\mathcal{W}_{2n}^+); \mathbb{R}((\hbar)))$$
$$\downarrow$$
$$\Omega^\bullet_X((\hbar))$$

This is the Gelfand-Fuks map.

· Characteristic classes

We review the Chern-Weil construction of characteristic classes in Lie alg. cohomology.

Let $\mathfrak{g}$ be a Lie alg. $\mathfrak{h} \subset \mathfrak{g}$ Lie subalgebra.

Let $\mathrm{Pr}: \mathfrak{g} \mapsto \mathfrak{h}$ $\mathfrak{h}$-equivariant splitting of the embedding $\mathfrak{h} \subset \mathfrak{g}$.

The failure of Pr being a Lie alg. homomorphism gives

$R \in \mathrm{Hom}(\Lambda^2 \mathfrak{g}, \mathfrak{h})$ by

$$R(\alpha, \beta) = [\mathrm{pr}(\alpha), \mathrm{pr}(\beta)]_{\mathfrak{h}} - \mathrm{pr}[\alpha, \beta]_{\mathfrak{g}}, \quad \alpha, \beta \in \mathfrak{g}.$$

$\mathfrak{h}$-equivariance implies that

$$R \in \mathrm{Hom}_{\mathfrak{h}}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{h})$$

R is called the curvature form.

Let $\mathrm{Sym}^m(\mathfrak{h}^\vee)^{\mathfrak{h}}$ be $\mathfrak{h}$-invariant polynomials on $\mathfrak{h}$ of deg $= m$. Given $P \in \mathrm{Sym}^m(\mathfrak{h}^\vee)^{\mathfrak{h}}$,

$$P(R) : \Lambda^{2m}\mathfrak{g} \xmapsto{\Lambda^m R} \mathrm{Sym}^m(\mathfrak{h}) \xrightarrow{P} \mathbb{R}$$

defines a cochain $P(R) \in C^{2m}(\mathfrak{g}, \mathfrak{h}; \mathbb{R})$

<u>Check</u>: $\partial_{Lie} P(R) = 0$

$$\Rightarrow [P(R)] \in H^{2m}(\mathfrak{g}, \mathfrak{h}; \mathbb{R})$$

Chern-Weil characteristic map

$$\chi : \mathrm{Sym}^\bullet(\mathfrak{h}^\vee)^{\mathfrak{h}} \longmapsto H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R})$$
$$P \longmapsto \chi(P) := [P(R)]$$

Now we consider the case

$$\boxed{\mathfrak{g} = W_{2n}^+, \quad \mathfrak{h} = \mathfrak{sp}_{2n} \oplus Z(\mathfrak{g})}$$

Given $f \in \mathcal{W}_{2n}^{+}$, $f = f(y^i, \hbar)$. $y^1 \cdots y^n = p_1, \cdots, p_n$

$y^{n+1} \cdots y^n = q^1, \cdots, q^n$

Define

$$\begin{cases} pr_1(f) = \frac{1}{2} \sum_{ij} \partial_i \partial_j f \Big|_{y=\hbar=0} y^i y^j \in sp_{2n} \\ pr_3(f) = f \Big|_{y=0} \in Z(g) \end{cases}$$

We obtain the corresponding curvature

$$\begin{cases} R_1 := [pr_1(-), pr_1(-)] - pr_1[-,-] & \in Hom(\Lambda^2 g, sp_{2n}) \\ R_3 := -pr_3[-,-] & \in Hom(\Lambda^2 g, \mathbb{R}[[\hbar]]) \end{cases}$$

Define the $\hat{A}$- genus

$$\hat{A}(sp_{2n}) = \left[\det \left(\frac{R_1/2}{\sinh(R_1/2)} \right)^{1/2} \right] \in H^{\cdot}(g, h; \mathbb{R})$$

Prop: Under desc: $H^{\cdot}(g, h; \mathbb{R}((\hbar))) \longmapsto H^{\cdot}(X)((\hbar))$ via the Fedosov connection,

$$\begin{cases} desc(\hat{A}(sp_{2n})) = \hat{A}(X) \\ desc(R_3) = \omega_\hbar - \hbar\omega \end{cases}$$

· Universal trace map

Recall that using $\Omega^{\bullet}_{S^1} \mapsto \mathbb{R}^{2n}$, we have obtained

$$\mathrm{Tr} = \int_{BU} \langle - \rangle_{free} : CC^{per}_{-\bullet}(\mathcal{W}_{2n}) \longmapsto \underbrace{\mathbb{R}((\hbar))[u,u^{-1}]}_{\mathbb{K}}$$

Let us write

$$\mathrm{Tr} \in \mathrm{Hom}_{\mathbb{K}}(CC^{per}_{-\bullet}(\mathcal{W}_{2n}), \mathbb{K})$$

this is a $(\overline{\mathcal{W}^+_{2n}}, Sp_{2n})$-module.

$\Big\downarrow$ via the flat $(\overline{\mathcal{W}}^+_{2n}, Sp_{2n})$-bundle $F_{Sp}(X) \to X$

Associated bundle

$$E^{per} := F_{Sp}(X) \underset{Sp_{2n}}{\times} \mathrm{Hom}_{\mathbb{K}}(CC^{per}_{-\bullet}(\mathcal{W}_{2n}), \mathbb{K})$$

w/. induced flat connection ∇^r.

Recall $\mathcal{W}(X) = F_{Sp}(X) \underset{Sp_{2n}}{\times} \mathcal{W}_{2n}$ w/ flat ∇^r.

We would like to glue Tr on X. Let us denote δ for the differential on $\mathrm{Hom}_{\mathbb{K}}(CC_{-\bullet}^{per}(\mathcal{W}_{2n}), \mathbb{K})$ induced from $b+uB$. So

$$\delta\, \mathrm{Tr} = \mathrm{Tr}((b+uB)(-)) = 0.$$

We can view Tr as defining an element in

$$C^{\bullet}_{Lie}(\mathfrak{g}, \mathfrak{h}; \mathrm{Hom}_{\mathbb{K}}(CC_{-\bullet}^{per}(\mathcal{W}_{2n}), \mathbb{K}))$$

where we take $\boxed{\mathfrak{g} = \mathcal{W}_{2n}^{+}/Z(\mathcal{W}_{2n}^{+}) \quad \mathfrak{h} = sp_{2n}.}$

However, Tr is not $\mathfrak{g}$-invariant, i.e.

$$\partial_{Lie}\, \mathrm{Tr} \neq 0.$$

In other words, Tr is NOT a map of $(\mathfrak{g}, Sp_{2n})$-module. So Tr can not be glued directly. It is observed that

$$\partial_{Lie}\, \mathrm{Tr} = \delta(-)$$

It turns out that we have a canonical way to lift Tr to

$$\widehat{\mathrm{Tr}} \in C^{\bullet}_{Lie}(\mathfrak{g}, \mathfrak{h}; \mathrm{Hom}_{\mathbb{K}}(CC^{per}_{-\bullet}(\mathcal{W}_{2n}), \mathbb{K}))$$

such that

$$\widehat{\mathrm{Tr}} = \mathrm{Tr} + \text{terms in } C^{>0}_{Lie}(\cdots)$$

and satisfying the coupled cocycle condition

$$(\partial_{Lie} + \delta)\,\widehat{\mathrm{Tr}} = 0.$$

$\widehat{\mathrm{Tr}}$ is called the "universal trace map"

Let us insert $1 \in \mathcal{W}_{2n}$, then

$\widehat{\mathrm{Tr}}(1)$ is ∂_{Lie}-closed, which defines the

universal index: $[\widehat{\mathrm{Tr}}(1)] \in H^{\bullet}_{Lie}(\mathfrak{g}, \mathfrak{h}; \mathbb{K})$

Thm [universal algebraic Index]

$$[\widehat{\mathrm{Tr}}(1)] = \hbar^{n} e^{-R_3/\hbar}\,\hat{A}(sp_{2n})_{\hbar}$$

where for $A = \sum_{p \text{ even}} A_p$, $A_p \in H^p(\mathfrak{g}, \mathfrak{h}; \mathbb{K})$, $A_{\hbar} = \sum_p \hbar^{-p/2} A_p$

- Feigin-Tsygan, Feigin-Felder-Shoikhet, Bressler-Nest-Tsygan ......

RK: This can be naturally generalized to the bundle case. See Gui-L-Xu.

Now we apply the Gelfand–Fuks descent

$$\hat{T}_r \qquad C^{\cdot}(g, h; \mathrm{Hom}_{\mathbb{K}}(CC^{per}_{-\cdot}(\mathcal{W}_{2n}), \mathbb{K}))$$

$$\Big\downarrow \mathrm{desc}$$

$$\Omega^{\cdot}(X, \mathrm{Hom}_{\mathbb{K}}(CC^{per}_{-\cdot}(W(X)), \mathbb{K})$$

Let $W_D(X) = \{\text{flat sections of } W(X)\}$ which gives a deformation quantization. Then

$$\mathrm{desc}(\hat{T}_r): \underset{b+uB}{CC^{per}_{-\cdot}(W_D(X))} \longmapsto \underset{d_X}{\Omega^{\cdot}(X)((\hbar))[u, u^{-1}]}$$

In particular, it defines a trace map in deformation quantization by

$$f \in W_D(X) \longmapsto \int_X \mathrm{desc}(\hat{T}_r)(f) \in \mathbb{R}((\hbar))$$

Can show this value does not involve u.

By the universal algebraic index theorem, we have

$$\int_X \mathrm{desc}(\widehat{\mathrm{Tr}})(1) = \int_X e^{-\omega_\hbar/\hbar}\hat{A}(X)$$

This gives the algebraic index theorem.

· The construction of $\widehat{\mathrm{Tr}}$

$$\begin{tikzcd} \boxed{\text{background symmetry}} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \boxed{\text{Connection form}} \arrow[dl, leftrightarrow] \\ & \boxed{\text{Interaction}} & \end{tikzcd}$$

$$C^\bullet(\mathfrak{g}, \mathfrak{h}; \mathrm{Hom}_{\mathbb{K}}(CC^{per}_{-\bullet}(\mathcal{W}_{2n}), \mathbb{K}))$$

Let $\Theta : \mathfrak{g} \longmapsto W_{2n}^+/Z(W_{2n}^+)$ $(=\mathfrak{g})$

be the canonical map. (identity)

For each $f \in W_{2n}^+/Z(W_{2n}^+)$, we have defined

the local functional on $\mathcal{E} = \Omega^\bullet(S^1)\otimes\mathbb{R}^{2n}$ by

$$I_f(\varphi) = \int_{S^1} f(\varphi) \qquad \varphi \in \mathcal{E}$$

Then Ⓗ gives a map

$$I_{Ⓗ} : \mathfrak{g} \longmapsto \mathcal{O}_{loc}(\mathcal{E}), \quad f \mapsto I_{Ⓗ(f)}$$

We can view this as

$$I_{Ⓗ} \in C^1(\mathfrak{g}, \mathcal{O}_{loc}(\mathcal{E})) = \mathfrak{g}^\vee \otimes \mathcal{O}_{loc}(\mathcal{E})$$

Now we can construct

$$\widehat{Tr} \in C^\bullet(\mathfrak{g}, \mathfrak{h}; \mathrm{Hom}_{\mathbb{K}}(CC^{per}_{-\bullet}(\mathcal{W}_{2n}), \mathbb{K})) \text{ by}$$

$$\widehat{Tr}(f_0 \otimes f_1 \otimes \cdots \otimes f_m) \qquad f_i \in \mathcal{W}_{2n}$$

$$:= \int_{BV} e^{\hbar P_0^\infty}\left(\mathcal{O}_{f_0, f_1, \cdots, f_m}\, e^{\frac{1}{\hbar} I_{Ⓗ}}\right) \in C^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{K})$$

$$“= \int_{BV} \int_{\substack{\mathrm{Im} d^* \\ \cap \\ \hat{\mathcal{E}}}} e^{-\frac{1}{2\hbar}\int_{S^1} \langle \varphi, d\varphi \rangle + \frac{1}{\hbar} I_{Ⓗ}}\, \mathcal{O}_{f_0, f_1, \cdots, f_m}”$$

- The computation of index

$\mathcal{W}_{2n}$ can be viewed as a family of associative algebras parametrized by $\hbar$.

$\rightsquigarrow$ $\nabla_{\hbar\partial_\hbar} \curvearrowright CC^{per}_{-\cdot}(\mathcal{W}_{2n})$

Getzler – Gauss – Manin connection

The calculation of index consists of the following steps

① Feynman Diagram Computation implies

$$\widehat{Tr}(1) = u^n e^{-R_3/u\hbar}\left(\hat{A}(sp_{2n})_u + O(\hbar)\right)$$

1-loop computation

② Computation of GGM connection shows

$$\nabla_{\hbar\partial_\hbar}\left(e^{R_3/u\hbar}\,\widehat{Tr}(1)\right) = \partial_{Lie}\text{-exact}$$

③ Combining ① and ②, we find

$$[\widehat{Tr}(1)] = \left[u^n e^{-R_3/u\hbar}\,\hat{A}(sp_{2n})_u\right] \in H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{K})$$